Hence, if E lies in the line DF,[1] the solution of the case of the problem we are considering is reduced to inscribing in the circle A a triangle $PQ'R'$, whose side produced shall pass through the fixed collinear points E, F, D'. This can be done by the lemma. The other cases may be treated in a similar way.

In the above I have neither paused to elaborate the details of the various steps nor endeavored to set forth the whole in the Greek manner. To fill in these lacunae, the interested reader may turn to the writings of Pappus, Simson, Flauti, Scorza, and Zeuthen[2]—for the weighty testimony of Zeuthen favors the solution just given as the most probable original of Apollonius.

In the course of this restoration not only have various properties of centers of similitude of circles, mentioned earlier in this paper, been used, but Monge's Theorem (A) has also been necessary. I therefore hold that Theorem (A) and centers of similitude were discovered by the Greeks. How much the more is the assertion concerning "Monge's Theorem" confirmed, when we recall that it follows immediately, on applying to the triangle ABC, the converse of the theorem of Menelaus of Alexandria (c. 80 A. D.) with regard to transversals.[3]

A CARDIOIDOGRAPH.

By C. M. HEBBERT, University of Illinois.

In his book called "Linkages"[4] J. D. C. DeRoos illustrates a rather complicated linkage for describing a cardioid. A simpler form of cardioidograph is presented in this paper.

The cardioid may be defined as the path traversed by any point of the circumference of a circle as it rolls upon a fixed coplanar circle of the same radius. In figure 1 consider the cardioid which is the path of the point B of the rolling

[1] Recalling the theorem of Aristarchus referred to above (note 8, page 9) a possible Greek mode of proof (followed by Monge) that D, E, F are collinear, is the following. Consider A, B, C as great circles of three spheres. If the two transverse planes tangent to the sphere A and to the spheres B and C, be drawn, these planes will each pass through the three centers of similitude, D, E, F; that is, these points lie on the line of intersection of the tangent planes. A second method of proof, in the Greek manner, is indicated below in note 3.

[2] H. G. ZEUTHEN, *Die Lehre von den Kegelschnitten im Altertum*, Deutsche Ausgabe besorgt von R. v. Fischer-Benson. Kopenhagen, 1886, pp. 381–383.

[3] In the above figure, for instance,

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{a}{b}\cdot -\frac{b}{c}\cdot\frac{c}{a}=-1,$$

and conversely. This is Lemma I, Book III, of the *Spherics* of Menelaus; or, to be more accurate, the lemma was stated in the form: If DEF is a transversal of the triangle ABC, the ratio of AF to FB is equal to the ratio compounded of the ratios CD to BD and CE to EA. (*Theodosii sphaericorum Lib. III . . . Menelai Sphaericorum Lib. III* . . . Messanae, 1558, pp. 36 *verso*—37 *recto* of second pagination, or p. 83 of *Menelai sphaericorum Libri III . . . curavit vir Cl. Ed. Halleius*. Oxonii, MDCCLVIII.)

[4] D. Van Nostrand, New York, 1879. English translation from the *Revue Universelle des Mines*.

circle. At the starting position shown by the dotted circle the point B was at A and the radius DB was in the position CA. Since arc AE = arc BE and the circles have equal radii, angle AOD = angle ODB. Also, $DB = \frac{1}{2}OD$. These two equalities are fundamental in the construction of the linkage.

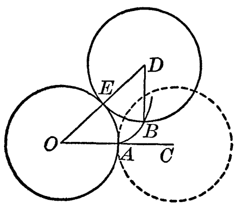

FIG. 1.

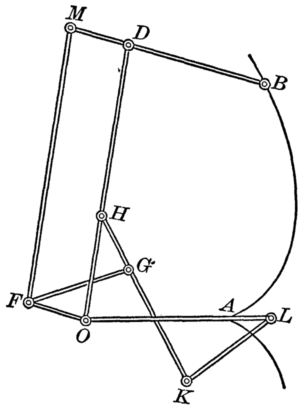

FIG. 2.

The links, BM, MF, DO, FG, HK, KL, OL, OF, are pinned together at D, M, F, G, H, K, L, and O, so that $OD = 2DB$; $DM = OF = HG$; $FM = OD$; $FG = OH = KL$; $HK = OL$; and $OH : HK = HG : OH$ (Fig. 2).

From the similar triangles OHK, OLK, OHG, and OFG, we have angle HOL = angle HOF.[1] In the parallelogram $OFMD$, angle FOD = angle ODB, i. e., angle AOD = angle ODB.

Hence, if the part OL is kept fixed the linkage compels the point B to move according to the two equalities derived from the definition of the cardioid, and a pencil fastened at B will describe a cardioid if OL is kept fixed while the other parts are free to move.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

One Thousand Exercises in Plane and Spherical Trigonometry. By EDWIN S. CRAWLEY. Published by E. S. CRAWLEY, University of Pennsylvania, Philadelphia, Pa., 1914, v+70 pages. \$0.50.

Professor Crawley, who is the author of a text-book on trigonometry, has recently published this book of exercises to be used with any text-book. The exercises are intended for class-room drill, and are therefore for the most part of an average grade of difficulty. The first 950 exercises are arranged in a classified list as follows: The measurement of angles (Ex. 1–25); The functions of one

[1] On the theory of linkages in general consult Arnold Emch: "Kinematische Gelenksysteme und die durch sie erzeugten geometrischen Transformationen," Solothurn, 1907.